SYNOPTIC: Eigenvalue Errors in the Method of Weighted Residuals, Jack G. Crouch, University of Dayton, Dayton, Ohio, William J. Anderson and Donald T. Greenwood, The University of Michigan, Ann Arbor, Mich.; *AIAA Journal*, Vol. 8, No. 11, pp. 2048–2054.

Structural Dynamic Analysis, Launch Vehicle and Controlled Missile Dynamics and Control

Theme

This paper presents a theoretical analysis of the error in approximate eigenvalues determined by the method of weighted residuals (MWR). The theoretical accuracy is compared with actual calculations of eigenvalue error in the problem of the lateral deflections of a large missile.

Content

The theoretical error analysis concerns a real, linear, eigenvalue problem of the form

$$L_{1x}(u) - \lambda L_{2x}(u) = 0 \qquad 0 \leq x \leq 1 \tag{1}$$

with $u(x)$ subject to the linear, homogeneous boundary conditions

$$B_s(u) = 0 \qquad s = 1,2,\ldots m \tag{2}$$

L_{1x} and L_{2x} are differential operators and m is the order of Eq. (1).

The approximate solution is of the form

$$\bar{u} = \sum_{i=l}^{n} c_i \phi_i \tag{3}$$

where $\{\phi_i(x);\ i = 1,2,\ldots n\}$ is a selected set of functions satisfying Eqs. (2). The coefficients $\{c_i;\ i = 1,2,\ldots n\}$ can be determined by requiring the residual error in the equation

$$\epsilon(\bar{u}) = L_{1x}(\bar{u}) - \lambda L_{2x}(\bar{u}) \neq 0 \tag{4}$$

to be orthogonal to each member of a set of weighting functions $\{w_i(x);\ i1,2,\ldots n\}$. Thus

$$\int_0^1 \epsilon w_i dx = 0 \qquad i = 1,2,\ldots n \tag{5}$$

The approximate eigenvalues are determined from the condition for solvability of Eqs. (5).

The approximating functions and the weighting functions differ from the exact eigenfunctions and adjoint eigenfunctions, respectively, of Eq. (1) by error functions proportional to a quantity η which is thought of as the order of magnitude of the error. If η is zero, the eigenvalues determined from Eqs. (5) will be the exact eigenvalues of Eq. (1).

The hypotheses are: a) Eq. (1) is a real, linear problem with real eigenvalues. b) The approximating functions possess convergent Fourier series in terms of the exact eigenfunctions. c) A system adjoint to Eqs. (1) and (2) exists whose boundary conditions do not involve the eigenvalue. d) The weighting functions possess convergent Fourier series in terms of the exact adjoint eigenfunctions. e) The approximate eigenvalues and eigenvectors appearing in the matrix form of Eqs. (5) are analytic functions of η.

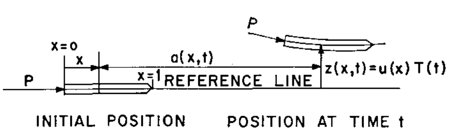

Fig. 1 Notation.

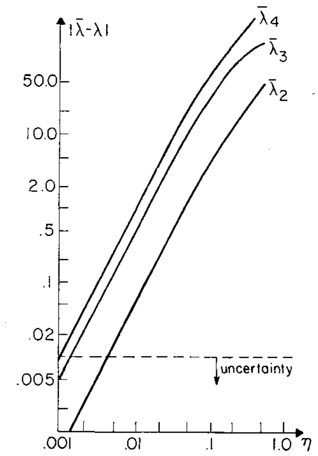

Fig. 2 Eigenvalue error.

The main theoretical result is that the error in the approximate eigenvalue is proportional to η^2 for small η. Thus

$$\bar{\lambda}_i = \lambda_i + \lambda_{i2}\eta^2 + \lambda_{i3}\eta^3 + \cdots \tag{6}$$

where $\bar{\lambda}_i$ is the approximate eigenvalue, λ_i is the exact eigenvalue and $\lambda_{i2}, \lambda_{i3}, \ldots$ are constants.

Earlier work has given the same result for the cases of self-adjoint problems and for the cases where the weighting functions satisfy the adjoint boundary conditions. The new feature of the present analysis is that the weighting functions may violate the adjoint boundary conditions by a small amount η.

A numerical example is solved to compare the eigenvalue error with the theoretical error stated above. Figure 1 illustrates the lateral deflection of a large missile with a thrust vector always tangent to the neutral axis. The thrust described renders the problem nonself-adjoint (nonconservative). The missile is treated as a uniform, "free-free" beam leading to the following nondimensionalized eigenvalue problem:

$$u'''' + \beta^2[(1 - x)u']' - \lambda u = 0 \tag{7}$$

$$u''' = u'' = 0 \text{ at } x = 0,1 \tag{8}$$

where $u(x)$ is the nondimensional, space dependent part of the lateral deflection $z(x,t)$. Equations (7) and (8) are solved for the approximate eigenvalues by MWR. The exact eigenvalues are obtained by other methods. Different sets of weighting functions are used to test their effects on eigenvalue accuracy.

From Eq. (6), with η small, one has approximately

$$\log |\bar{\lambda}_i - \lambda_i| = 2 \log\eta + \log |\lambda_{i2}| \tag{9}$$

Figure 2 shows typical results. The weighting functions used here do not satisfy the adjoint boundary conditions. The logarithmic eigenvalue error has a slope of two, agreeing with the theory.